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By Order of the Council,
ARTHUR H. BAILEY,
ARCHIBALD DAY,
Honorary Secretaries.

12, St. James's Square, S.W.
1*st June*, 1867.

On the Construction of Tables by the Method of Differences. By Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

Section IV.—*On the Construction of Tables in which the Characteristic Function is Irrational.*

(118). Irrational Functions, strictly speaking, are functions whose numerical values cannot be expressed by finite fractions. In the present connexion the term irrational has a wider signification. It is used to denote any function whose numerical value cannot, in general, be expressed *exactly* within the limits as to decimal places, to which we restrict ourselves in the table under formation. As here employed therefore the term designates not only transcendental functions, (as exponential, logarithmic, circular, &c.,) and algebraical irrational functions, (which are such as contain fractional powers of the variable in either numerator or denominator, or both,) but also algebraical fractional functions, which are such as contain integer powers of the variable in both numerator and denominator, or in the latter only. And it is to the formation of tables of the values of functions such as these that our attention is now to be directed.

(119). The characteristic of irrational functions which renders necessary a different mode of treatment from that which we found so efficient when the functions dealt with were rational, is that those functions have no constant differences. It was shown (60) that in a rational function of the nth degree the nth difference is constant, so that those of the $(n+1)$th and all higher orders vanish. The consequence of this with which we are here concerned is that where a single value of a function of the nth degree, and its differences up to the nth inclusive are given, we can readily form by means of them, and continue to any extent desired, a series of successive values of the function. Now irrational functions when expanded take the form of infinite series: their degree is therefore infinite, and consequently their differences never vanish. It is obvious then that we cannot in regard to them adopt the same course as in the case of rational functions. It is conceivable however, and it will presently be shown to be the fact, that, neglecting all beyond, we may assume a difference, of an order depending on the specialties of the case, to be constant for a definite interval, without producing sensible error in the values formed upon this hypothesis. The operation of construction will thus become assimilated in a measure to that with which we are already familiar, the chief point of distinction being that here we shall have to form a new series of differences corresponding to every change that we make in the value of the final difference, which is treated, for a certain interval, as constant.

(120). In forming a table of the values of an irrational function our basis of operations, so to speak, is a series of equidistant values of the function, the intervals being greater than those that are to be between the values constituting the table. Those values, which we call the fundamental or primitive values, may either be given, or may have to be formed by aid of the characteristic function, when it is known; and from the differences derived from them we form, by application of the formulæ in (78), the differences pertaining to the series to be constructed. After this the operation proceeds pretty much as in the case of rational functions, with, however, certain modifications, which will be exhibited and explained in due course.

(121). It is a necessary preliminary in our operations that the differences of the series of primitive values should converge, that is, that as they ascend in order they should decrease in magnitude; and it is obvious that the more rapidly they converge, the fewer orders of differences will it be necessary to take account of, as we shall thus the sooner reach an order small enough to be neglected. When the characteristic function is given we have it in our power usually to obtain the requisite degree of convergence in the differences, by forming primitive values sufficiently close. That the differences are more convergent for small intervals than for large will appear by inspection of the formulæ in (78). Thus, if of two series (commencing at the same point) of equidistant values of the same function, the interval between successive terms of the first be m times that between successive terms of the second, then, if the differences of the first be

$$\Delta^1, \Delta^2, \Delta^3 \ldots \Delta^r,$$

those of the second will be, *approximately*,

$$\frac{\Delta^1}{m}, \frac{\Delta^2}{m^2}, \frac{\Delta^3}{m^3} \ldots \frac{\Delta^r}{m^r}$$

respectively. And these, m being greater than unity, are obviously more convergent than the others.

(122). It will have been already perceived, and it will become more apparent hereafter, that the labour attending the use of a considerable number of orders of differences in construction is not small; and it increases in a rapidly increasing ratio, as that number is increased. There are two methods of avoiding, as far as may be, the labour attending the use of many orders of differences. One of them was hinted at in (121); and it consists in forming fundamental values so close that the *final* difference it is necessary to take account of shall not be of a high order. This method is available up to a certain point; that is till we have got the orders of differences it is necessary to take account of compressed into a manageable number—say four, five or six, according to the extent of the table to be formed. To go beyond this would be to incur a portion of the very labour which the use of the method of differences is designed to supersede. And therefore, when this point has been reached we have recourse to the other method to which reference has been made, and which I now proceed to describe.

(123). Suppose that a table has to be constructed comprising 100,000 values of a given function; and suppose also that, having formed each thousandth value, we find on differencing the one hundred values so formed, that the fifth order of differences is so nearly constant that those of the sixth and higher orders may be neglected. It might appear that we could now proceed to the completion of the table by the insertion of 999 terms in each interval. We certainly could; but there are objections all but insuperable to such a course. The construction of 100,000 terms by the use of five orders of differences would, in the most favourable circumstances, be a sufficiently formidable undertaking, while here the circumstances are about as unfavourable as they could well be. The cause of this is the comparative magnitude of the intervals in the fundamental series and in the series to be formed. The ratio is 1000 : 1; hence to form the differences to be used in the construction, *m*, in the formulæ of (78), will have to be made 1000, and it is apparent that, as a consequence, they would extend seventeen places at least to the right of the last place in the fundamental values. It is not easy to form a just estimate of the immense mass of figures, useless in the final results, that we should thus be compelled to write. Add to this that we should have no verification of our work as it proceeds, but on the attainment of each thousandth term of the series being formed, and it is quite plain that nothing further need be said to show the desirableness of avoiding, if possible, this way of going to work.

(124). The remedy for these inconveniences is to effect the required subdivision of the intervals in the fundamental series by *successive* operations. Thus, in each interval, instead of 999, I should first insert 9 terms; in each interval of this new series I should again insert 9 terms, and so on till the table should be completed. By conducting the process in this way the number of *extra* figures is very much reduced, as is also the number of orders of differences necessary to be taken account of. In the first interpolation only, comprising 1000 terms, should we have to use so many as five orders; for the second, (10,000 terms,) four—perhaps three—would suffice; and for the third, (100,000) two orders would be sufficient; and this last, it will be found, can be made a very simple operation. In all these operations also, it will be perceived that we have a very effectual check on the formation of every tenth term.

(125). I have *said* that in certain circumstances *small* differences may be neglected without producing sensible effect on the resulting series. It is time now to *show* in what circumstances this may be done; and also *how small* the differences must be in order that it may be done with safety.

(126). The expression for the *n*th term of a series whose initial term is u_x is (70),

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{1.2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x+\ldots,$$

the interval between successive terms being unity, to be afterwards subdivided by giving to *n* fractional values (76).

The contribution of a specific difference, say the *r*th, to u_{x+n} is,

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}\Delta^r u_x,$$

which for convenience we may write

$$\frac{\phi(n)}{1.2\,.\,.\,.\,r}\Delta^r u_x.$$

Now the greatest error committed by neglecting this difference will be in that term or in those terms corresponding to that value or those values of n which give to $\phi(n)$ the greatest *numerical* value, whether positive or negative. Such value or values of n will evidently have to be sought amongst those which render $\phi(n)$ a maximum or a minimum;* and we shall therefore have to examine $\phi(n)$ with a view to determine its maxima and minima, for the integer values of r from 2 to 7, the sixth being the highest order of differences we have made provision for using.

(127). To determine the values of the variable which render an algebraical function of the rth degree a maximum or a minimum we require generally to solve an equation of the $(r-1)$th degree. This would be rather a tedious and troublesome operation in the case of values of r so great as some of those we have to deal with. It is fortunate therefore that here, owing to the peculiar form of $\phi(n)$, we are able to effect our object by the solution of an equation in each case of not more than half the dimensions of that which, but for this peculiarity of form, would have been necessary.

(128). The property of which advantage is thus taken may be enunciated as follows:—

If $\phi(n)$ be a function of n of the form

$$n(n-1)(n-2)\,.\,.\,.\,.\,(n-r+1),$$

the substitution in it for n of $\frac{1}{2}(r-1)+a$, (a being any positive number, rational or irrational,) will give the same result *numerically* as the substitution of $\frac{1}{2}(r-1)-a$; and the signs of the results will be the same or different according as n is an even or an odd number.†

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* It may be necessary to explain that a maximum is not necessarily the greater nor a minimum the least value that a function can assume. A function of the nth degree has usually $n-1$ maxima and minima values; and their characteristics are that a maximum is greater than preceding and following values, and a minimum is less. Thus, if a be a value of x that renders $\phi(x)$ a maximum, we must have $\phi(a)$ *greater* than both $\phi(a+h)$ and $\phi(a-h)$; and if a renders $\phi(x)$ a minimum we must have $\phi(a)$ *less* than both $\phi(a+h)$ and $\phi(a-h)$, h being in both cases supposed indefinitely small. It is obvious then, that it is *amongst* the maxima and minima values of a function that the greatest numerical value it can assume must be sought.

† The above is a case of the following more general theorem:—If $\phi(x)$ be a function of the nth degree whose roots are

$$0, t, 2t, 3t\,.\,.\,.\,.\,(n-1)t,$$

and if $k=\frac{1}{2}(n-1)t$, substitution in $\phi(x)$ of $k+a$ and $k-a$ successively, for x (a being any number, rational or irrational), will give results equal in numerical value, and of the same or different signs, according as n is an even or an odd number. That is, we shall always have, in a function constituted as above,

$$\phi(k+a)=(-1)^n\phi(k-a).$$

This theorem, which I do not recollect to have met with anywhere, is very easily proved.

(129). I do not here offer any demonstration of this property, but content myself with giving one or two numerical illustrations. Thus, if r (which is the number of factors, and consequently denotes the degree of $\phi(n)$), be 5, then $\frac{1}{2}(r-1)=2$; and if we take 6 for a, we have

$$\phi(2+6) \text{ or } \phi(8) \;=\; 8\,.\,7\,.\,6\,.\,5\,.\,4$$
$$\phi(2-6) \text{ or } \phi(-4)=-4.-5.-6.-7.-8$$

Again, r still being 5, let $a=\frac{3}{4}$.

$$\therefore \phi(2+\tfrac{3}{4}) \text{ or } \phi(2\tfrac{3}{4})=2\tfrac{3}{4}\,.\,1\tfrac{3}{4}\,.\,\tfrac{3}{4}.-\tfrac{1}{4}.-1\tfrac{1}{4}$$
$$\phi(2-\tfrac{3}{4}) \text{ or } \phi(1\tfrac{1}{4})=1\tfrac{1}{4}\,.\,\tfrac{1}{4}.-\tfrac{3}{4}.-1\tfrac{3}{4}.-2\tfrac{3}{4}.$$

It is needless to perform the multiplications here indicated. The two sets of factors in each example are the same numerically, and so also consequently will be the products. These moreover will differ in sign, r being in both examples an odd number. If in like manner an example be formed in which r is an even number, the numerical identity of the factors for any value of a, and consequently of the products, will still subsist; and these last will be in this case of the same sign.

(130). The manner in which the property just explained is made available for the reduction of the degree of the equation to be solved is as follows:—If, writing k for $\frac{1}{2}(r-1)$, $k+a$, $k+\beta$, $k+\gamma$, &c. are values of n which correspond to maxima or minima values of $\phi(n)$, so also are $k-a$, $k-\beta$, $k-\gamma$, &c.; since, as has just been shown, the results of the substitution in $\phi(n)$ of the latter set of values are the same *numerically* as those of the substitution of the former. But, by the Theory of Equations, the values of n which render $\phi(n)$ a maximum or a minimum are the roots of the *derived function* of $\phi(n)$ (the manner of forming which will be presently shown); hence this function, which is denoted by $\phi'(n)$, will admit of being written

$$\phi'(n)=\mathrm{A}\{(n-\overline{k+a})(n-\overline{k-a})(n-\overline{k+\beta})(n-\overline{k-\beta})\;.\;.\;.\;.\;\},$$

in which A is the coefficient of the highest power of n. The last expression may also obviously be written thus,

$$\phi'(n)=\mathrm{A}\{(\overline{n-k}-a)(\overline{n-k}+a)(\overline{n-k}-\beta)(\overline{n-k}+\beta)\;.\;.\;.\;.\;\};$$

from which it appears that if we transform $\phi'(n)$ into $\phi'(n+k)$ we shall obtain

$$\phi'(n+k)=\mathrm{A}\{(n-a)(n+a)(n-\beta)(n+\beta)\;.\;.\;.\;.\;\},$$

an expression whose roots are obviously $+a$, $-a$, $+\beta$, $-\beta$, &c.

If moreover we multiply the factors of this expression in pairs it will become

$$\phi'(n+k)=\mathrm{A}\{(n^2-a^2)(n^2-\beta^2)\ldots\};$$

and hence it appears that, whatever be the degree of $\phi'(n)$, if we transform it into $\phi'(n+k)$, k being known and equal to $\frac{1}{2}(r-1)$, (where r is the number expressing the degree of $\phi(n)$,) it must and it will assume *the form* of a function of half the degree of $\phi'(n)$, and whose roots are a^2, β^2, &c. And these roots being found those of $\phi'(n)$ are known.

(131). It will be observed that the roots of $\phi'(n)$ go in pairs, $k\pm a$, $k\pm\beta$, &c. This is intelligible when $r-1$, the degree of $\phi'(n)$, is an even number; but it is less so, at first sight, when $r-1$ is an odd number. The explanation is that in the latter case one of the quantities a, β, &c., say a, will vanish, and the pair, $k\pm a$, will coalesce into a single root, namely k.

(132). We are now prepared to enter upon an examination of the terms of the expansion of u_{x+n} in detail. I commence with the term involving Δ^7u_x. This term is

$$\frac{\phi(n)}{1.2\ldots7}\Delta^7u_x, \text{ or } \frac{\phi(n)}{5040}\Delta^7u_x,$$

in which

$$\phi(n)=n(n-1)(n-2)\ldots(n-6);$$

and we have first to find the values of n which render this function a maximum or a minimum. The required values are, as stated (130), the roots of the derived function of $\phi(n)$. The derived function is formed as follows:—Multiplying out the factors of $\phi(n)$ we have

$$\phi(n)=n^7-21n^6+175n^5-735n^4+1624n^3-1764n^2+720n.$$

Now multiply each term of this by the exponent of n in the same term, and diminish the exponent in the result by unity. The aggregate of the terms so formed is the derived function.* Thus n^7 gives $7n^6$, $-21n^6$ gives $-126n^5$, and so on. We hence have

$$\begin{aligned}\phi'(n)&=7n^6-126n^5+875n^4-2940n^3+4872n^2-3528n+720\\&=7(n^6-18n^5+125n^4-420n^3+696n^2-504n+102\cdot857143).\end{aligned}$$

Now r being here equal to 7, $k=\frac{1}{2}(r-1)=3$. Therefore transform $\phi'(n)$ into $\phi'(n+3)$, as follows :—

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* The function which in the Theory of Equations receives the name of the *derived function* of $\phi(n)$, is that which in the Higher Analysis is known as the *differential coefficient* of $\phi(n)$.

	1	−18	125	−420	696	−504	102·857143(3
		−15	80	−180	156	−36	−5·142857
		−12	44	−48	12	0	
		−9	17	+3	21		
		−6	−1	0			
		−3	−10				
		0					
Result,	1	*	−10	*	+21	*	−5·142857

That is,

$$\phi'(n+3)=7(n^6-10n^4+21n^2-5\cdot142857).$$

(133). This function may obviously be considered as of the third degree in n^2, and its roots, a^2, β^2, &c. evolved accordingly (130). I give the evolution of one of them, the greatest, say a^2, leaving the others as exercises for the student.

1	−10	21	−5·142,857(7·1718856
	−3	0*	−5·142
	+4	2800	−2·231857
	110	2911	−60044
	111	302300	−28206
	112	310259	−2726
	1130	$31826\cancel{7}$	−178
	1137	31838	−19
	1144	$3185\cancel{0}$	0
	$\cancel{1}1\cancel{5}1$		

The three roots are,

$a^2=7\cdot171886$ $\quad\therefore a=\pm2\cdot67804$
$\beta^2=2\cdot546521$ $\quad\beta=\pm1\cdot59578$
$\gamma^2=\cdot281594$ $\quad\gamma=\pm\cdot53065$

And the roots of $\phi'(n)$, $k\pm a$, $k\pm\beta$, &c., are consequently, arranging in order of magnitude and omitting superfluous figures,

·322	3·531
1·404	4·596
2·469	5·678

(134). The above roots of $\phi'(n)$ are, as stated (130), the values of n which correspond to maxima and minima values of $\phi(n)$, and we have to determine which of these maxima and minima are the greatest numerically, positive or negative. For this purpose we substitute the roots of $\phi'(n)$ in succession in $\phi(n)$, or rather the first three of them, since we know that the second three will give the same results as the first, in reverse order, and with opposite signs, r being here an odd number (128).

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* What we learn from the occurrence of cipher here is that 7, the transforming number, is a root of the quadratic whose coefficients are those of the first three terms of $\phi'(n+3)$, viz., $n^2-10x+21$. The other root is obviously $10-7=21\div7$, $=3$. These however are matters with which we are not particularly concerned here.

The results are as follows, in order:—

95·84	−12·36
−23·15	23·15
12·36	−95·84

So that the greatest positive value of $\phi(n)$ is, say, 96, corresponding to $n = {\cdot}322$, and the greatest negative value is -96, which corresponds to $n = 5{\cdot}678$.

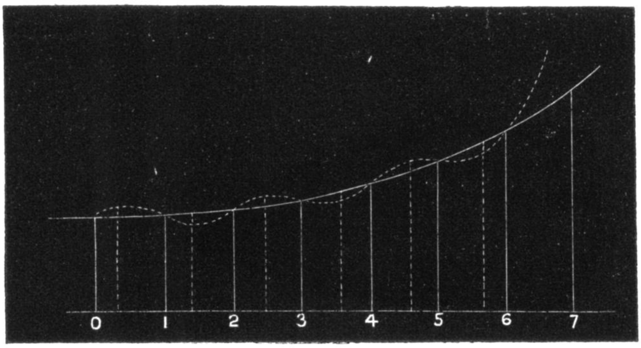
(135). The term involving $\Delta^7 u_x$ is (132),

$$\frac{\phi(n)}{5040}\Delta^7 u_x;$$

substituting in which the last named values of $\phi(n)$, we have for the greatest error arising from the neglect of $\Delta^7 u_x$,

$$e = \pm\frac{96}{5040}\Delta^7 u_x = \pm{\cdot}01905\Delta^7 u_x.$$

(136). Perhaps the foregoing investigation and its results will be rendered more intelligible by a geometrical illustration.



Let the *continuous* curve in the adjoining figure be the *locus* of the equation

$$y = u_{x+n},$$

where u_{x+n} is of the form

$$u_x + n\Delta u_x + \frac{n(n-1)}{2}\Delta^2 u_x + \ldots\ldots + \frac{n(n-1)(n-2)\ldots(n-6)}{1.2\ldots7}\Delta^7 u_x,$$

terms involving $\Delta^8 u_x$, &c., being supposed too small to have any sensible influence between the limits $n=0$ and $n=6$. That is, any value of n being measured from 0 (the *origin*) along the *axis*, an *ordinate* erected from its extremity, meeting the curve, is the corresponding value of y. Thus, the ordinates drawn from the points 0, 1, 2, &c., will represent u_x, u_{x+1}, u_{x+2}, &c., the terms from which the differences, Δu_x, &c., are derived. Now the investigation shows that if the curve be drawn corresponding to the same equation, omitting the term involving $\Delta^7 u_x$, it will be *of the character* shown by the waved and dotted curve. Since the omitted term vanishes for integer values of n from 0 to 6, inclusive, the two curves will intersect at the points whose *abscissæ* have these values; but between those points the second curve will deviate from the first alternately on the one side and the other, since for points situated in adjoining intervals the coefficient of $\Delta^7 u_x$ is alternately a maximum and a minimum (133). I have indicated by the dotted ordinates the position of the points in the curve where the maxima and minima deviations occur. They are those corresponding to the values of n in (133), namely ·322, 1·404, &c., one in each interval. The portions of these ordinates intercepted between the two curves, are the measures of the deviations at the points in question; and it is shown in (134) that the two points of greatest deviation are those corresponding to $n = {\cdot}322$, and $n = 5{\cdot}678$,* the deviation at the former point being positive and that at the latter negative, supposing $\Delta^7 u_x$ positive. Finally I have determined in (135) the magnitude of the deviation at the points in question.

(137). We have now to investigate the effect of neglecting $\Delta^6 u_x$, differences of the seventh and higher orders being here supposed to be too small to be taken account of between the limits $n=0$, and $n=5$.

The term involving $\Delta^6 u_x$ is

$$\frac{\phi(n)}{1.2\ldots6}\Delta^6 u_x, \text{ or } \frac{\phi(n)}{720}\Delta^6 u_x,$$

in which

$$\phi(n) = n(n-1)(n-2)\ldots(n-5)$$
$$= n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n;$$

and the derived function is, (132),

$$\phi'(n) = 6n^5 - 75n^4 + 340n^3 - 675n^2 + 548n - 120.$$

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* This would be apparent from the figure were it correctly drawn, and on a sufficiently large scale. But it is not correctly drawn, the dotted curve being sketched by hand. And the deviations *in the figure* are not necessarily greatest at the points indicated. Also, from the necessity of the case, the dotted curve is so drawn that the portions of it situated between the points of intersection with the continuous curve are alternately concave and convex to the axis of abscissæ, in consequence of which the points in question become *points of contrary flexure*. Neither is this necessarily the case. It will only be so when the neglected term in u_{x+n} is comparatively large. It is easy to conceive, though very difficult to draw, a curve which, while lying alternately on opposite sides of the given curve, shall yet, like it, be convex to the axis throughout.

Now here $r=6$. Hence $k=\frac{1}{2}(r-1)=2\cdot5$, and we transform $\phi'(n)$ into $\phi'(n+2\cdot5)$ thus :—

	6	−75	340	−675	548	−120(2·5
		−60	190	−200	48	0
		−45	77·5	−6·25	32·375	
		−30	2 5	0		
		−15	−35			
		0				
Result,	6	*	−35	*	32·375	*

That is,

$$\phi'(n+2\cdot5)=6n^5-35n^3+32\cdot375n,$$

a root of which is obviously 0, say $\alpha^2=0$, (131) ; and the remaining roots are those of

$$6n^4-35n^2+32\cdot375,$$

which may be treated as a quadratic in n^2, as follows :—

6	−35	$3\dot{2}\cdot3\dot{7}5\dot{0}$(1·1528323
	−29	3·37
	−230	1·1350
	−224	60000
	−2180	17624
	−2150	683
	−21200	48
	−21188	6
	$-2117\cancel{6}$	

The other root may be either developed in the same way, or got by subtracting that just found from $35\div6$, (the quotient of the second coefficient by the first, with its sign changed,) $=5\cdot8333333$.

(138). We thus have for the roots of $\phi'(n+2\cdot5)$,

$$\begin{array}{ll} \alpha^2=0 & \therefore\ \alpha=\pm0 \\ \beta^2=1\cdot152832 & \beta=\pm1\cdot07494 \\ \gamma^2=4\cdot680501 & \gamma=\pm2\cdot16344; \end{array}$$

and the roots of $\phi'(n)$, being $k\pm\alpha$, $k\pm\beta$, &c., are consequently,

$$\begin{array}{lll} \cdot337 & 2\cdot500 & 3\cdot575 \\ 1\cdot425 & & 4\cdot663 \end{array}$$

These are the values of n which correspond to maxima and minima values of $\phi(n)$; and it will be remarked that, like those determined in (133), they fall very near the middle of the intervals which separate the given values, u_x, u_{x+1}, &c. Their places, were the intervals divided into ten parts, would be in the neighbourhood of the 3rd, 14th, 25th, 36th, and 47th terms.

(139). Now substituting the above values successively in $\phi(n)$ we get,

−16·90	−3·52	5·05
5·05		−16·90

Hence the term involving $\Delta^6 u_x$ being, (137),

$$\frac{\phi(n)}{720}\Delta^6 u_x,$$

the greatest value it can assume, and consequently the greatest error committed by neglecting this term will be,

$$e = -\frac{16{\cdot}90}{720}\Delta^6 u_x = -{\cdot}02347\Delta^6 u_x.$$

(140). The term involving $\Delta^5 u_x$ is,

$$\frac{\phi(n)}{1.2\ldots5}\Delta^5 u_x, \text{ or } \frac{\phi(n)}{120}\Delta^5 u_x,$$

where $\phi(n) = n(n-1)(n-2)\ldots.(n-4),$

$$= n^5 - 10n^4 + 35n^3 - 50n^2 + 24n.$$

The derived function is, (132),

$$\phi'(n) = 5n^4 - 40n^3 + 105n^2 - 100n + 24,$$

or $\phi'(n) = 5(n^4 - 8n^3 + 21n^2 - 20n + 4{\cdot}8).$

Here r being equal to 5, k or $\frac{1}{2}(r-1) = 2$. Therefore transform $\phi'(n)$ into $\phi'(n+2)$ as follows:—

	1	−8	21	−20	4·8(2
		−6	9	−2	·8
		−4	1	0	
		−2	−3		
		0			
Result,	1	*	−3	*	·8

Hence, $\phi'(n+2) = 5(n^4 - 3n^2 + {\cdot}8);$

and this may be treated as a quadratic in n^2, as follows:—

1	−30*	80(·29584054
	−28	2400
	−260	14100
	−251	202500
	−2420	9764
	−2415	130
	−24100	10
	−24092	
	$-2408\cancel{4}$	

The other root will be got by a similar development, or by subtracting that just found from 3, the second coefficient with its sign changed.

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* The roots are multiplied by 10 at the outset, the first root figure here being in the first decimal place (32).

(141). We thus have,

$$\begin{array}{ll} \alpha^2 = \cdot 29584 & \therefore \alpha = \pm \cdot 54391 \\ \beta^2 = 2\cdot 70416 & \quad\; \beta = \pm 1\cdot 64444 \end{array}$$

And the roots of $\phi'(n)$, being $k \pm \alpha$, $k \pm \beta$, are consequently,

$$\begin{array}{rr} \cdot 356 & 2\cdot 544 \\ 1\cdot 456 & 3\cdot 644 \end{array}$$

(142). Now substituting these values successively in $\phi(n)$ we get,

$$\begin{array}{rr} 3\cdot 63 & 1\cdot 42 \\ -1\cdot 42 & -3\cdot 63 \end{array}$$

Hence the term involving $\Delta^5 u_x$ being (140),

$$\frac{\phi(n)}{120} \Delta^5 u_x,$$

the greatest error consequent on the omission of this term will be,

$$e = \pm \frac{3\cdot 63}{120} \Delta^5 u_x = \pm \cdot 03025 \Delta^5 u_x.$$

(143). The term involving $\Delta^4 u_x$ is

$$\frac{\phi(n)}{1.2.3.4} \Delta^4 u_x, \text{ or } \frac{\phi(n)}{24} \Delta^4 u_x,$$

in which

$$\begin{aligned} \phi(n) &= n(n-1)(n-2)(n-3) \\ &= n^4 - 6n^3 + 11n^2 - 6n. \end{aligned}$$

The derived function is (132),

$$\begin{aligned} \phi'(n) &= 4n^3 - 18n^2 + 22n - 6 \\ &= 2(2n^3 - 9n^2 + 11n - 3). \end{aligned}$$

Here $r=4$, and k or $\frac{1}{2}(r-1) = 1\cdot 5$. Transform $\phi'(n)$ into $\phi'(n+1\cdot 5)$ as follows:—

$$\begin{array}{rrrrl} & 2 & -9 & 11 & -3(1\cdot 5 \\ & & -6 & 2 & 0 \\ & & -3 & -2\cdot 5 & \\ & & 0 & & \\ \hline \text{Result,} & 2 & * & -2\cdot 5 & * \end{array}$$

That is, $\quad \phi'(n+1\cdot 5) = 2(2n^3 - 2\cdot 5n),$

a root of which is $\alpha^2 = 0$, and the remaining roots are contained in $2n^2 - 2\cdot 5$, or $n^2 - 1\cdot 25$. This gives,

$$\beta^2 = 1\cdot 25 \quad \therefore \beta = \pm 1\cdot 1180.$$

(144). Hence the roots of $\phi'(n)$, $k \pm \alpha$, $k \pm \beta$, are

$$\begin{array}{r} \cdot382 \\ 1\cdot500 \\ 2\cdot618 \end{array}$$

And substituting these in $\phi(n)$ we obtain,

$$\begin{array}{r} -1\cdot00 \\ 0\cdot45 \\ -1\cdot00 \end{array}$$

Hence the greatest error committed by neglecting a fourth difference will be

$$e = -\frac{1}{24}\Delta^4 u_x = -\cdot04167\Delta^4 u_x.$$

(145). The term involving $\Delta^3 u_x$ is,

$$\frac{\phi(n)}{1.2.3}\Delta^3 u_x, \text{ or } \frac{\phi(n)}{6}\Delta^3 u_x,$$

in which
$$\phi(n) = n(n-1)(n-2)$$
$$= n^3 - 3n^2 + 2n;$$

and the derived function of this is, (132),

$$\phi'(n) = 3n^2 - 6n + 2.$$

Here $r = 3$, and $k = \frac{1}{2}(r-1) = 1$. Transform therefore $\phi'(n)$ into $\phi'(n+1)$ thus:—

$$\begin{array}{lccc} & 3 & -6 & 2(1 \\ & & -3 & -1 \\ & & 0 & \\ \hline \text{Result,} & 3 & * & -1 \end{array}$$

That is,
$$\phi'(n+1) = 3n^2 - 1;$$

which gives,
$$\alpha^2 = \frac{1}{3} \quad \therefore \alpha = \pm\cdot57735.$$

(146). Hence the roots of $\phi'(n)$ are

$$\cdot4226 \text{ and } 1\cdot5774;$$

and substituting these in $\phi(n)$ we have

$$\cdot385 \text{ and } -\cdot385.$$

Hence the greatest error committed by neglecting $\Delta^3 u_x$ will be,

$$e = \pm\frac{\cdot385}{6}\Delta^3 u_x = \pm\cdot06417\Delta^3 u_x.$$

(147). The term involving $\Delta^2 u_x$ is,

$$\frac{\phi(n)}{2}\Delta^2 u_x,$$

where
$$\phi(n) = n(n-1) = n^2 - n.$$

The derived function of this is, (132),

$$\phi'(n)=2n-1,$$

which needs no transformation, but gives at once the root of $\phi'(n)=\cdot 5$. And substituting this for n in $\phi(n)$ we get,

$$\phi(\cdot 5)=-\cdot 25.$$

Hence the term involving $\Delta^2 u_x$ becomes,

$$e=-\frac{\cdot 25}{2}\,\Delta^2 u_x=-\cdot 125\Delta^2 u_x.$$

(148). Collecting our results, we have found for e, the value of the greatest error committed by neglecting a difference of each order from the second to the seventh, as follows:—

$$\begin{aligned} e&=\cdot 1250\Delta^2 u_x, & e&=.0303\Delta^5 u_x,\\ e&=\cdot 0642\Delta^3 u_x, & e&=.0235\Delta^6 u_x,\\ e&=\cdot 0417\Delta^4 u_x, & e&=\cdot 0191\Delta^7 u_x. \end{aligned}$$

(149). But these results will be more convenient for use if put in the following form. Solving the several equations for $\Delta^2 u_x$, &c., we get

$$\begin{aligned} \Delta^2 u_x&=8e, & \Delta^5 u_x&=33e,\\ \Delta^3 u_x&=16e, & \Delta^6 u_x&=43e,\\ \Delta^4 u_x&=24e, & \Delta^7 u_x&=52e. \end{aligned}$$

It is now apparent that, having determined on the greatest error that can be tolerated in the values to be formed, we can, by substituting that error in the expression for any difference that may be in question, at once learn whether that difference may be safely neglected or not. Thus if, for example, an error of unit in the last place of the values to be formed be admissible, and if the differences of the sixth and higher orders are inappreciable, then the fifth difference also may be neglected if it do not exceed 33. And similarly of the other differences in the table.

(150). There is another source of error in the interpolation of irrational functions. The fundamental values are not absolutely correct; they are correct only to the nearest unit in the last place. In consequence the interpolated values may, and occasionally will, err by as much as a unit in the last place. If moreover these values, with their possible errors, are used as fundamental values for further interpolation, it is obvious that any errors in them will be reproduced, in a somewhat aggravated form, in the new values. A very simple and efficient remedy exists for this. It is to carry our computations to one, two, or three places* (according to the degree of accuracy required) beyond the number that are to be retained and tabulated. The effect is, that in computing to two places beyond the number required for tabulation, our knowledge of the possibility of an error of say 3, in the last of them, will only leave us in doubt, in the few cases in which the extra figures are between 47 and 53, whether the usual conventional increase is due to the last figure retained or not. And if we go to three extra places the doubtful cases, for a possible error of 3, will be between the limits 497 and 503. Hence if the facility for the occurrence of the possible error for all points of the series were the same, we should probably have in the first case, three instances in every hundred of the values formed, and in the second three in every thousand, in which the increase is wrongly given or withheld. Such an amount of liability to error as this will not disturb the equanimity of either the most timid computer, or the greatest stickler for accuracy.

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* These extra places are irrespective of, and in addition to, those arising in consequence of the fractional values of the several differences in (81).

(151). The preliminary matter being now disposed of, we are ready to enter upon examples of the actual construction of tables. The first example will be a table of Anti-logarithms, that is, of the numbers corresponding to all logarithms from 0 to 1.

The Mutual Life Insurance Company of New York.

ON the 1st February, 1868, this Company, "now the largest moneyed Corporation" in America, celebrated its "silver wedding," as it was termed by the Governor of the State of New York on the occasion of the 25th annual meeting. The Report then presented and the general Prospectus of the Company have been forwarded to us by the "President" of the Company, who appears to be virtually what in this country we should call "Managing Director." These documents are so instructive and interesting in themselves, and illustrate so remarkably the application to our pursuits of that originality of the American character, which is fast becoming as proverbial as its energy, that we propose to notice their contents in some detail.

Though most of us have a vague notion that the operations of American Insurance Companies are conducted on a scale of great magnitude, few will learn without astonishment that this particular company alone issued, in its last financial year, no fewer than 19,406 Policies, for sums amounting in the aggregate to over